Malaysian Journal of Science **16B**: 3-6 (1995)

Analytical expressions for the nuclear elastic scattering phase shifts in heavy ion collisions

Bernardine R. Wong
Department of Mathematics, University of Malaya, 59100 Kuala Lumpur, Malaysia

ABSTRACT The elastic scattering of heavy ions is studied within the semiclassical framework. Analytical expressions are obtained for the elastic scattering phase shifts due to the nuclear potential using the one-turning-point Wentzel-Kramers-Brillouin approximation. The analytical expressions yield good agreement with respect to results obtained numerically through fully quantal calculations.

ABSTRAK Sebaran kenyal ion berat dikaji di bawah rangka separuhklasik. Ungkapan analitik dihasil untuk anjakan fasa sebaran kenyal daripada keupayaan nukleus dengan menggunakan kirahampir satu-titik-beluk Wentzel-Kramers-Brillouin. Ungkapan analitik memberi keputusan yang bersetuju dengan kiraan berangka kuantum mekanik.

(elastic scattering, heavy–ion collisions)

INTRODUCTION

The collisions of heavy ions, i.e. ions whose masses are heavier than that of an α - particle, have been studied theoretically and experimentally for over thirty years [1] in an effort to understand the structure of nuclei and the interactions between them. Experiments have shown that the interaction between heavy ions can be approximated by a complex optical model potential. The imaginary part of this potential simulates absorption, i.e. the loss of flux from the elastic scattering channel into the inelastic and reaction channels.

In elastic scattering, the quantities of interest are the elastic scattering phase shifts $\delta\,(l)$ which are expressed in terms of the partial waves l. These phase shifts are extracted in a fully quantum mechanical calculation by numerically solving the time-independent Schrödinger equation to obtain the wavefunction that represents the relative motion of the heavy ions. The wavefunction thus obtained and its first derivative are then matched onto the corresponding Coulomb functions at a suitably large radius where the nuclear potentials become negligible. This procedure is repeated over a large range of partial waves, yielding the elastic scattering phase shifts. Although the numerical evaluation of such phase shifts can nowadays be routinely performed with available computer codes, it is useful to develop analytical expressions for the phase shifts so that the characteristics of the interaction potential may be extracted.

THE ANALYTICAL NUCLEAR ELASTIC SCATTERING PHASE SHIFTS

An analytical expression for the elastic scattering phase shifts may be derived using the fact that the wavelength associated with the relative motion of heavy ions is generally much smaller than the dimensions of the ion-ion interaction system. This permits the use of a semiclassical description. One semiclassical approach [2,3] is via the Wentzel-Kramers-Brillouin (WKB) approximation [4], which yields the nuclear elastic scattering phase shift as the difference between two integrals [5]. These integrals evaluate the phase shift

$$\delta(l) = \int_{r_0}^{\infty} k(r)dr - \int_{r_c}^{\infty} k_c(r)dr. \qquad (1)$$

Eq. (1) expresses the nuclear elastic phase shift in terms of the difference in the absolute phase, calculated by the integral. Each integral results in a phase. The first integral involves the radial wavenumber *k(r)* which is given by

$$k(r) = \{2\mu(\frac{h}{2\pi})^{-2}\,[E - V_N(r) - V_C(r) - V_l(r)]\}^{1/2} \qquad (2)$$

In Eq. (2), μ and E are respectively the reduced mass and the kinetic energy for the heavy ion system, and h is the Planck constant. The subscripts N, C and l respectively denote the nuclear, Coulomb and the centrifugal terms of the interaction potential. r_0 is the turning point at which the wavenumber vanishes. The second integral in Eq. (1) differs from the first in that the nuclear term V_N is set equal to zero.

The Coulomb potential felt by the heavy ions of charge (mass) numbers Z_1 (A_1) and Z_2 (A_2) is given by

$$V_C(r) = \frac{Z_1Z_2e^2}{2R_C}[3-(\frac{r}{R_C})^2], \quad r \leq R_C$$

$$V_C(r) = Z_1Z_2e^2/r, \quad r \geq R_C \tag{3}$$

where e is the electronic charge and R_C is the range of the Coulomb potential. The centrifugal potential is described by

$$V_l(r) = l(l+1)(\frac{h}{2\pi})^2/(2\mu r^2), \tag{4}$$

where the reduced mass of the heavy ions is given by $\mu = A_1 A_2 / (A_1 + A_2)$.

The nuclear potential comprises the real and imaginary contributions such that

$$V_N(r) = V(r) + iW(r). \tag{5}$$

The specific shape for the nuclear potential used in this study is that of Hill and Ford [6] which has the form

$$V(r) = -V_0\{2 - \exp[(r-R_R)/a_R]\}, \quad r \leq \mathrm{R_R}$$

$$V(r) = -V_0 \exp[-(r-R_R)/a_R], \quad r \geq \mathrm{R_R} \tag{6}$$

Here, V_0, R_R and a_R are the well depth, range and diffuseness parameters for the real nuclear potential. The imaginary nuclear potential *W(r)* possesses the same form as Eq. (6) with W_0, R_I and a_I being the corresponding parameters.

Numerical calculations using Eq. (1) have yielded close agreement with fully quantal calculations [2,3] when the integrals are evaluated in the complex r-plane along a suitable integration path. Brink [7] simplified the evaluation of Eq. (1) by making the following approximations: (i) the nuclear potential is regarded as being very weak compared with the overall potential, allowing the turning points r_0 and r_C to become coincident; (ii) the relative motion of the heavy ions is assumed to follow a straight path, whereby the main contribution to Eq. (1) arises when the ions are closest together. By using the above approximations in Eq. (1), Brink [7] expressed the real part of the nuclear elastic scattering phase shift in terms of the perturbation formula

$$\delta_R^0(l) = -\mu(\frac{h}{2\pi})^{-2}\int_{r_C}^{\infty} V(r)\frac{dr}{k_C(r)}$$

$$= 2\frac{V_0 d}{hv}\exp(R_R/a_R)\, K_1(d/a_R), \quad d \geq R_R. \tag{7}$$

Here, v is the relative tangential velocity [8] of the heavy ion system at the distance of closest approach, d. The relative tangential velocity is obtained semi-classically as

$$v = \frac{h}{2\pi}(l+\frac{1}{2})/(\mu d), \tag{8}$$

while the distance of closest approach of the ions is given by

$$d = \{\eta + [\eta^2 + (l+\frac{1}{2})^2]^{1/2}\}/\kappa. \tag{9}$$

Here, η and κ respectively denote the Sommerfeld parameter and the asymptotic wavenumber. Eq. (7) is valid for the partial waves whose distance of closest approach is larger than the dimensions of the ion-ion system. K_1 is the modified Bessel function of order unity [9]. The corresponding expression for the imaginary nuclear elastic scattering phase shift is

$$\delta_I^0(l) = -\mu(\frac{h}{2\pi})^{-2}\int_{r_C}^{\infty} W(r)\frac{dr}{k_C(r)}$$

$$= 2\pi\frac{W_0 d}{hv}\exp(R_I/a_I)\, K_1(d/a_I), \quad d \geq R_I \tag{10}$$

Eqs. (7) and (10) imply that the real part of the nuclear potential, *V(r)*, is solely responsible for the real nuclear elastic scattering phase shift. Similarly, the imaginary part of the nuclear potential alone generates the imaginary nuclear elastic scattering phase shifts, giving rise to absorption. This is a consequence of the approximations made by Brink [7].

In this paper, we demonstrate that if the use of the approximation $r_0 = r_C$ is avoided, the real and the imaginary parts of the nuclear potential will both contribute to the scattering as well as to the absorption in the elastic scattering process. We follow the prescription of Brink and Satchler [10] who transform the range coordinate *r(s)* according to

$$V_N(r) + V_C(r) + V_l(r) = V_C(s) + V_l(s), \tag{11}$$

so that $k(r) = k_C(s)$. In this manner, the elastic scattering phase shift (containing both the real and the imaginary parts) can be expressed [10] as

$$\delta(l) = \mu(\frac{h}{2\pi})^{-2} \int_{r_c}^{\infty} [r(s) - s]$$
$$[V_C'(s) + V_I'(s)] \frac{ds}{k_C(s)} \qquad (12)$$

In Eq. (11), the prime indicates the derivative with respect to s. If a Taylor expansion is performed on the potential terms up to first order, an approximation to the $[\ r(s) - s\ ]$ term is obtained. This results in the elastic scattering phase shift being expressed as

$$\delta(l) = \mu(\frac{h}{2\pi})^{-2} \int_{r_c}^{\infty} V_N(s) \{1 - V_N'(s) /$$
$$[V_C'(s) + V_I'(s) + V_N'(s)]\} \frac{ds}{k_C(s)} . \qquad (13)$$

A comparison between eqs. (13) and (7) indicates that eq. (13) contains an additional term which involves the ratio of the derivatives of the potentials. This factor includes, to first order, the effect of allowing r_0 to differ from r_C. An analytical expression for the elastic scattering phase shift may be obtained from Eq. (13) if it is observed that the the main contribution of the ratio of derivatives to the integral arises when $s = r_C$. Thus, the expressions for the real and the imaginary parts of the elastic scattering phase shift may be written as

$$\delta_R(l) = \gamma_1(d)\, \delta_R^0(l) + \gamma_2(d)\, \delta_I^0(l) \qquad (14a)$$

$$\delta_I(l) = \gamma_3(d)\, \delta_I^0(l) + \gamma_4(d)\, \delta_R^0(l). \qquad (14b)$$

Here, the multiplicative factors γ are functions of the distance of closest approach d and are defined as

$$\gamma_1 = 1 - \{\tfrac{1}{\sqrt{2}}(V_C' + V_I')V_N'$$
$$-[\tfrac{1}{\sqrt{3}}(V_N')^2 + \sqrt{\tfrac{a_I}{a_I+2a_R}}\ (W')^2]\}/(V_{CIN}')^2$$

$$\gamma_2 = [\tfrac{1}{\sqrt{2}}(V_C' + V_I')W']/(V_{CIN}')^2$$

$$\gamma_3 = 1 - \{\sqrt{\tfrac{a_R}{a_R+a_I}}\ (V_C' + V_I')V_N'$$
$$-[\sqrt{\tfrac{a_R}{a_R+2a_I}}\ (V_N')^2 + \tfrac{1}{\sqrt{3}}(W')^2]\}/(V_{CIN}')^2$$

$$\gamma_4 = -[\sqrt{\tfrac{a_I}{a_I+a_R}}\ (V_C' + V_I')W']/(V_{CIN}')^2. \qquad (15)$$

Eqs. (14) and (15) are valid when the distance of closest approach exceeds the smaller of the two radii R_R and R_I. In Eq. (15),

$$(V'_{CIN})^2 = (V_C' + V_I' + V_N')^2 + (W')^2. \qquad (16)$$

COMPARISON WITH NUMERICAL COMPUTATIONS

A comparison is made of the elastic scattering phase shifts obtained using Eqs. (7), (10) and (14a,b) with the phase shifts obtained via a fully quantum mechanical calculation for the elastic scattering of ^{16}O and ^{208}Pb at 192 MeV incident laboratory energy. The nuclear potential parameters V_0 = 40 MeV , W_0 = 35 MeV, $R_R = R_I = R_C$ = 10.35 fm, $a_R = a_I$ = 0.634 fm, were obtained from Ball *et al.* [11]. Fig. 1 shows the real part of the elastic scattering phase shift versus partial wave, while Fig. 2 displays the variation of the imaginary part of the phase shift. Both figures show that the analytical formula of Brink [7], i.e. Eqs. (7) and (10), agree closely with the numerical results for partial waves greater than 110. This illustrates that the approximations of Brink [7] are valid at sufficiently large values of partial wave. On the other hand, at partial waves between 90 and 110, the predictions of Eqs. (7) and (10) deviate substantially from the numerical results. However, the use of Eqs. (14a,b) result in a significant improvement in this range of partial waves. This implies that the approximation $r_0 = r_C$ becomes invalid in this region. For partial waves less than 90, both sets of analytical formulas become invalid as the distance of closest approach is less than the radius of the heavy ion system and the nuclear potential can no longer be treated perturbatively.

CONCLUSION

In this paper, analytical expressions for the real and imaginary parts of the elastic scattering phase shifts were developed under less restrictive approximations than those considered by Brink [7]. These expressions were shown to remain valid over a broader range of partial waves, especially in the region where the heavy ions just begin to overlap. In addition,

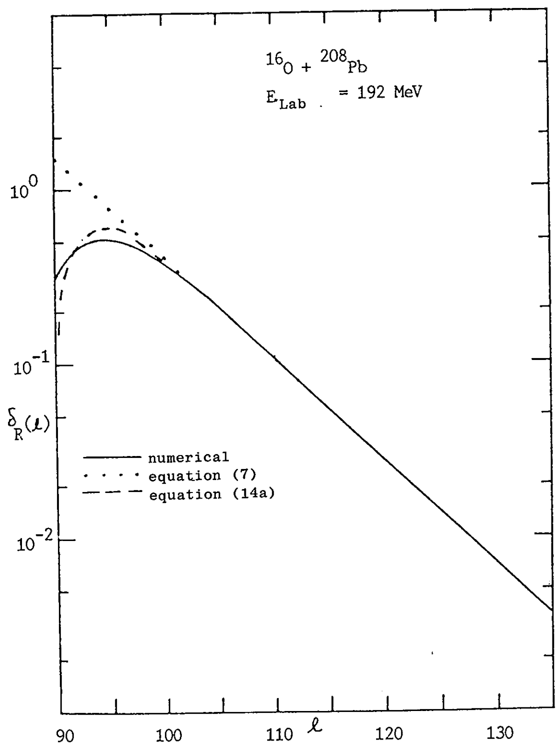

Figure 1. The real part of the elastic scattering phase shift versus partial wave.

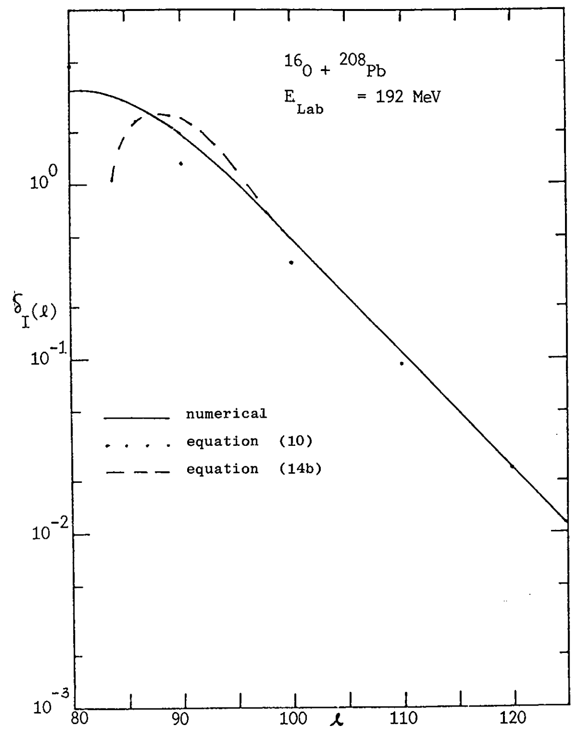

Figure 2. The imaginary part of the elastic scattering phase shift versus partial wave.

these expressions show that the real and the imaginary parts of the optical model potential contribute toward both the scattering as well as the absorptive parts of the elastic scattering phase shifts.

Acknowledgments The author gratefully acknowledges the support of the University of Malaya (F407/94 and F407/95).

REFERENCES

1 Bass R. (1980) *Nuclear Reactions with Heavy Ions*, Berlin, Springer-Verlag.

2 Landowne S., Dasso C.H., Nilsson B.S., Broglia R.A. and Winther A. (1976) *Nucl. Phys.* **A259**: 99.

3 Knoll J. and Schaeffer R. (1976) *Ann. Phys.* **97**: 307.

4 Wentzel G. (1926) *Z. Phys.* **38**: 518; Kramers H.A. (1926) *Z. Phys.* **38**: 828; Brillouin L. (1926) *C.R. Acad. Sci. Paris* **183**: 24.

5 Landau L.D. and Lifschitz E.M. (1962) *Quantum Mechanics*, Oxford, Pergamon.

6 Hill D.L. and Ford K.W. (1954) *Phys. Rev.* **94**: 1617.

7 Brink D.M. (1978) In: *Nuclear Physics with Heavy Ions and Mesons*, ed. Balian R., Rho R. and Ripka G. New York, North-Holland.

8 Wong B.R. and Low K.S. (1990) *J. Phys.* **G16**: 841.

9 Abramowitz M. and Stegun I.A. (1964) *Handbook of Mathematical Functions* Washington D.C., Dover.

10 Brink D.M. and Satchler G.R. (1981) *J. Phys.* **G7**: 43.

11 Ball J.B., Fulmer C.B., Gross E.E., Halbert M.L., Hensley D.C., Ludemann C.A., Saltmarsh M.J. and Satchler G.R. (1975) *Nucl. Phys.* **A252**: 208.